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INTEGRABILITY OF G-STRUCTURES AND FLAT MANIFOLDS

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Following the terminology set out by Chern [8], [9], a *G-structure* on an n-dimensional smooth manifold M is a fixed reduction of the structural group of the tangent bundle for M to the subgroup G of the general linear group $Gl(n, \mathbf{R})$ on $\mathbf{R}^n$. We say that a G-structure on M is *integrable*, if the reduction can be realized by an atlas, (cf. §1).

The existence of an integrable G-structure on a smooth manifold imposes in general strong conditions on the topology of the manifold. As our guiding example we mention the extreme case, where we deal with a parallelization (G is the trivial group). Here the complete answer is as follows.

Theorem [10]. *A connected n-dimensional smooth manifold M is integrably parallelizable if and only if either* (i) *M is noncompact and parallelizable, or* (ii) *M is diffeomorphic to the n-dimensional torus T^n.*

The purpose of the present paper is primarily to investigate integrable G-structures for the class of finite subgroups in $Gl(n, \mathbf{R})$. We prove in particular

Theorem A. *A compact n-dimensional smooth manifold M with $n \geqslant 2$ admits an integrable G-structure with G a finite group if and only if it admits a flat Riemannian structure.*

In order to handle noncompact manifolds in an equally satisfying way as compact manifolds, we have found it necessary to extend our investigations to include integrable G-structures for the class of totally disconnected (discrete) subgroups in $Gl(n, \mathbf{R})$. In doing this we obtain the following theorem, which phrased in a different language essentially can be found in Auslander and Markus [5].

Theorem B. *An n-dimensional smooth manifold M with $n \geqslant 2$ admits an integrable G-structure with G a discrete subgroup in* $Gl(n, \mathbf{R})$ *if and only if it admits an affine flat structure.*

As an introduction to the investigation of integrable G-structures, we prove in Theorem 1.1 a result of independent interest, which can be seen as a

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generalization of the construction of a holonomy covering space by Auslander and Markus [5] and further treated by Auslander [2]. Theorem 1.1 states roughly that if G is an arbitrary subgroup of $Gl(n, \mathbf{R})$ with identity subgroup G_0, then corresponding to an integrable G-structure we can obtain an integrable G_0-structure by passing to a suitable principal bundle space with fibre G/G_0. In particular, the manifolds in Theorem A and Theorem B are therefore covered by integrably parallelizable manifolds.

In the last section we point out how results on flat Riemannian and affine manifolds by Bieberbach, Auslander and Kuranishi can be used to give information on integrable G-structures. In particular we prove (Theorem 3.3) that for an arbitrary finite group G there exists a compact manifold with an integrable G-structure, which does not admit an integrable finite H-structure unless G is isomorphic to a subgroup of H. On the other hand we also like to think of our results as differential topological characterizations of affine flat (resp. Riemannian flat) manifolds.

Finally, we thank the referee for valuable comments and for pointing out Example 3.2.

1. A partial reduction to connected groups

Throughout this paper M denotes a connected paracompact n-dimensional smooth manifold with $n \geqslant 2$, and G denotes a subgroup of the general linear group $Gl(n, \mathbf{R})$ on $\mathbf{R}^n$, with identity component G_0. Note that G_0 is a normal subgroup of G, and that the quotient group G/G_0 is discrete.

An atlas $\{(U_i, \varphi_i)\}$ for the smooth structure on M is called a *G-atlas*, if all differentials in the overlap between charts belong to G. The existence of an integrable G-structure on M is then equivalent to the existence of a G-atlas on M.

Given a G-structure on the manifold M, and let $P(M) \to M$ denote the corresponding principal G-bundle over M. By dividing out the action of G_0 on $P(M)$ we obtain a principal G/G_0-bundle $P(M)/G_0 \to M$. Note that $P(M)/G_0$ has an induced G_0-structure and is a covering space over M. We shall now prove the corresponding statement for integrable G-structures.

Theorem 1.1. *Suppose the manifold M admits an integrable G-structure. Then there exists a principal G/G_0-bundle π: $\overline{M} \to M$ such that $\overline{M}$ admits an integrable G_0-structure.*

Proof. Let $\{(U_i, \varphi_i)\}$ be a G-atlas on M indexed by $i \in J$, and let $\amalg_{i \in J}(U_i \times G/G_0)$ denote the disjoint union of the spaces $U_i \times G/G_0$. Let $[g]$ denote the element in G/G_0 represented by $g \in G$, and introduce an equivalence relation on $\amalg_{i\in J}(U_i \times G/G_0)$ by defining $(x_i, [g_i]) \in U_i \times G/G_0$ to be equivalent to $(x_j, [g_j]) \in U_j \times G/G_0$ if and only if $x_i = x_j$ and $[D(\varphi_j \circ \varphi_i^{-1})_{\varphi_i(x_i)}] \cdot [g_i] = [g_j]$. Let $\overline{M}$ denote the quotient space, and q: $\amalg_{i\in J}(U_i \times G/G_0) \to \overline{M}$ the quotient map. When G/G_0 is given the discrete topology, and $\overline{M}$ the quotient topology, it is well known that the natural projection π: $\overline{M} \to M$ is a principal G/G_0-bundle.

Put $U_{i,g} = q(U_i \times [g])$ for $i \in J$ and $g \in G$. Since G/G_0 is discrete, $U_{i,g}$ is an open subset of $\overline{M}$. Let $\varphi_{i,g}$: $U_{i,g} \to \mathbf{R}^n$ be the unique map, which makes the following diagram commutative

$$\begin{array}{ccccc} U_i & \hookrightarrow & U_i \times [g] & \xrightarrow{q} & U_{i,g} \\ & \searrow^{g^{-1}\circ\varphi_i} & & & \downarrow \varphi_{i,g} \\ & & & & \mathbf{R}^n. \end{array}$$

Clearly $\{(U_{i,g}, \varphi_{i,g})\}$ is an atlas for the smooth structure on $\overline{M}$ with index set $J \times G$. Our claim is that it is in fact a G_0-atlas on $\overline{M}$. To prove this, assume that $U_{i,g} \cap U_{j,h} \neq \emptyset$, and let $z = q(x_i, [g]) = q(x_j, [h])$ be an arbitrary point in $U_{i,g} \cap U_{j,h}$. Then $x_i = x_j$ and

$$\left[D\left(\varphi_j \circ \varphi_i^{-1}\right)_{\varphi_i(x_i)}\right] \cdot [g] = [h],$$

or equivalently

$$h^{-1} \circ D\left(\varphi_j \circ \varphi_i^{-1}\right)_{\varphi_i(x_i)} \circ g \in G_0.$$

Since $\varphi_{j,h} \circ \varphi_{i,g}^{-1} = \varphi_{j,h} \circ q \circ \varphi_i^{-1} \circ g = h^{-1} \circ (\varphi_j \circ \varphi_i^{-1}) \circ g$, and h^{-1} and g are linear maps, we get

$$D\left(\varphi_{j,h} \circ \varphi_{i,g}^{-1}\right)_{\varphi_{i,g}(z)} = h^{-1} \circ D\left(\varphi_j \circ \varphi_i^{-1}\right)_{\varphi_i(x)} \circ g.$$

This differential belongs to G_0, and therefore $\{(U_{i,g}, \varphi_{i,g})\}$ is a G_0-atlas on $\overline{M}$ as asserted. q.e.d.

If G is a totally disconnected group, i.e., $G_0 = \{1\}$, we get

Corollary 1.2. *Suppose M admits an integrable G-structure for a discrete subgroup G in $Gl(n, \mathbf{R})$. Then there exists a principal G-bundle π: $\overline{M} \to M$, for which $\overline{M}$ is integrably parallelizable.*

By taking a connected component in the manifold $\overline{M}$ in Corollary 1.2 and using the main result in [10], we obtain in particular the following.

Corollary 1.3. *Suppose M is compact and admits an integrable G-structure for a finite subgroup G in $Gl(n, \mathbf{R})$. Then there exist a subgroup H in G and a principal H-bundle π: $\overline{M} \to M$, where $\overline{M}$ is the n-torus T^n.*

On the other hand, in Example 3.5 we exhibit a principal $\mathbf{Z}_k$-bundle π: $\overline{M} \to M$, where $\overline{M}$ is the torus and M does not admit an integrable finite

G-structure. The point is that not only do we have a principal H-bundle $\pi: \overline{M} \to M$ in Corollary 1.3, but in fact H preserves a natural geometric structure on $\overline{M}$ inherited from M.

2. Integrability of discrete and finite G-structures

In this section we prove Theorem A and Theorem B in the introduction. For expositional reasons we prove Theorem B first.

Recall that an affine flat manifold M is a smooth manifold with a linear connection ∇ whose curvature tensor and torsion tensor vanish. It is well known that a manifold is affine flat if and only if it is locally affine diffeomorphic to the affine space A^n.

Theorem B is a consequence of Proposition 2.1 and Proposition 2.2 below.

Proposition 2.1. *Let G be a totally disconnected subgroup of $Gl(n, \mathbf{R})$, and suppose that M admits an integrable G-structure. Then M admits an affine flat structure with a subgroup Φ of G as holonomy group.*

Proof. Let $\{(U_i, \varphi_i)\}$ be a G-atlas for M with G a totally disconnected group. We claim that the affine flat structures on the open sets U_i induced from the affine flat space A^n by the coordinate maps $\varphi_i: U_i \to A^n$ agree on nonempty intersections $U_i \cap U_j$, and hence define an affine flat structure on M. In order to prove this let $c: I \to U_i \cap U_j$ be a smooth curve, and $X: I \to TM$ a smooth vector field along c. Then X is parallel along c with respect to the connection defined by φ_i if and only if $\varphi_{i_*} \circ X = (\varphi_i \circ c, X_i)$ is parallel along $\varphi_i \circ c$ in A^n, i.e., if and only if $(d/dt)X_i = 0$. Similarly with i replaced by j. Now $X_j(t) = D(\varphi_j \circ \varphi_i^{-1})_{\varphi_i(c_i(t))}(X_i(t))$ by definition of X_i and X_j. Since G is totally disconnected and I is connected, $D(\varphi_j \circ \varphi_i^{-1})_{\varphi_i(c(t))} \in G$ is a fixed linear map of $\mathbf{R}^n$ and therefore $(d/dt)X_i = 0$ if and only if $(d/dt)X_j = 0$. This proves that the two connections introduced on $U_i \cap U_j$ are identical as claimed. By covering a piecewise smooth loop in M with a finite chain of coordinate domains U_i, it is easy to see that any element in the holonomy group Φ is a finite composition of differentials of coordinate changes and hence belongs to G. This completes the proof of Proposition 2.1. q.e.d.

It is well known, and not difficult to prove, that the holonomy group Φ of an affine flat manifold M is totally disconnected, see e.g., [12] or [14]. The converse to Proposition 2.1 is therefore a consequence of the following.

Proposition 2.2. *An affine flat manifold M is integrablely reducible to its holonomy group Φ.*

Proof. Let $\{U_i\}_{i\in J}$ be an open covering of the affine flat manifold M by convex sets, and pick arbitrary points $p_i \in U_i$. Fix one such point p_{i_0} and choose piecewise smooth curves c_i joining p_{i_0} and p_i for each $i \neq i_0$. Let τ_i denote the parallel translation from p_{i_0} to p_i along c_i. Define an atlas $\{(U_i, \varphi_i)\}$ on M by taking as charts $\varphi_i\colon U_i \to T_{p_{i_0}}M \cong \mathbf{R}^n$, where $\varphi_i = \tau_i^{-1} \circ \exp_{p_i}^{-1}$, and $\exp_{p_i}$ denotes the exponential map at p_i defined by the given connection. We claim that $\{(U_i, \varphi_i)\}$ is a Φ-atlas. In fact, since M has vanishing curvature and torsion, it follows easily from the Jacobi equation that $D(\varphi_j \circ \varphi_i^{-1})_{\varphi_i(x)} = \tau_j^{-1} \circ \tau_{jx}^{-1} \circ \tau_{ix} \circ \tau_i$ for any $x \in U_i \cap U_j$. Here τ_{ix} denotes the parallel translation from p_i to x along the unique geodesic from p_i to x in U_i. By definition of the holonomy group Φ we thus have $D(\varphi_j \circ \varphi_i^{-1}) \in \Phi$ as claimed. q.e.d.

Note that if M is a flat Riemannian manifold in Proposition 2.2, then the holonomy group Φ is actually a totally disconnected subgroup of the orthogonal group $O(n)$.

Suppose now on the other hand that G is contained in a compact subgroup G' of $Gl(n, \mathbf{R})$. By averaging the Euclidean inner product on $\mathbf{R}^n$ over G' if necessary, we can assume that G is a subgroup of $O(n)$. If G is a subgroup of $O(n)$, an argument similar to, but even simpler than, the proof of Proposition 2.1 shows that a G-atlas on a manifold M defines a flat Riemannian structure on M. This proves

Proposition 2.3. *If M admits an integrable G-structure with G contained in a compact subgroup of $Gl(n, \mathbf{R})$, then M admits a flat Riemannian structure with a totally disconnected subgroup Φ of G as holonomy group.*

It is well known that the holonomy group of a compact flat Riemannian manifold is finite. This is a consequence of Bieberbach's structure theorem for crystallographic groups [7]; cf. also [3], [12] or [14].

Theorem A in the introduction is therefore an immediate consequence of the following corollary to Proposition 2.2 and Proposition 2.3.

Corollary 2.4. *A manifold M admits an integrable finite G-structure if and only if it admits a flat Riemannian structure with finite holonomy group Φ.*

In general the holonomy group of a noncompact flat Riemannian manifold is infinite (cf. [12]). However, we do not know the answer to

Question 2.5. Do there exist flat Riemannian manifolds (complete or not), for which every flat Riemannian structure has infinite holonomy group?

Note that it is relatively easy to prove that a compact manifold with an integrable finite G-structure is covered by a torus (Corollary 1.3). The corresponding statement for a compact flat Riemannian manifold is equivalent to the deep structure theorem for crystallographic groups proved by Bieberbach [7].

3. Concluding remarks

The purpose of this section is to point out some interesting consequences of the results in §2 from the point of view of integrable G-structures.

From the work of Auslander [1] and Auslander-Markus [6] we know that there are infinitely many different homotopy types of compact affine flat (in fact Lorentzian flat) manifolds of dimension 3. On the other hand, we know from the work of Bieberbach [7] (cf. also Auslander [3] or Wolf [14]), that homotopy equivalent compact flat Riemannian manifolds are in fact affine equivalent, and that there are only finitely many compact flat Riemannian manifolds in each dimension up to affine equivalence. In dimension 3 we could also refer to the classification by Wolf [14]. These facts together with Theorem A and Theorem B in the introduction establish

Theorem 3.1. *There exist compact manifolds, which admit an integrable discrete G-structure but no integrable finite G-structure.*

The following concrete example of such a manifold was pointed out by the referee.

Example 3.2. Let $M = N_3/\Gamma$ be the homogeneous space, where N_3 is the group of real 3×3-matrices of the form

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \quad x, y, z \in \mathbf{R},$$

and Γ is the subgroup of N_3 consisting of matrices with integral entries.

Clearly M is compact and admits an integrable Γ-structure. Suppose that M admits an integrable G-structure for G a finite group. Then by Corollary 1.3, it is covered by the 3-torus, and hence the fundamental group Γ of M contains a free abelian subgroup of rank 3. This is a contradiction, since an easy computation shows that the centralizer of an arbitrary nontrivial element of Γ is free abelian of rank 2.

In the same spirit as Theorem 3.1 we have

Theorem 3.3. *Let G be any finite group. Then there exists a compact connected manifold M, which admits an integrable G-structure such that if M also admits an integrable H-structure for a finite group H, then G is isomorphic to a subgroup of H.*

Proof. The first statement follows from Proposition 2.2 together with the theorem of Auslander-Kuranishi [4] that there exists a compact flat Riemannian manifold M with holonomy group G. Suppose this manifold M admits an integrable H-structure with H a finite group. From Proposition 2.3 it then follows that M admits a flat Riemannian structure with a subgroup Φ of H as holonomy group. However, homotopy equivalent compact flat Riemannian manifolds are affine equivalent [7] (cf. also [3] or [14]), and thus in particular they have isomorphic holonomy groups. Consequently $G \cong \Phi$. Since Φ is a subgroup of H, Theorem 3.3 follows. q.e.d.

Proceeding along the lines in the proof of Theorem 3.3. we also easily get

Theorem 3.4. *Let M be a compact smooth manifold which admits an integrable G-structure for a finite group G. Then G can be so chosen that if M also admits an integrable H-structure for a finite group H, then G is isomorphic to a subgroup of H.*

We finish the paper by constructing an example which shows that the converse to Corollary 1.3 does not hold.

Example 3.5. Let Σ^n be a homotopy sphere of dimension n, and consider the connected sum $M = T^n \# \Sigma^n$. When Σ^n is not the standard sphere, M is P.L. homeomorphic but not diffeomorphic to T^n; cf. Wall [13, §15A]. On the other hand, any compact flat Riemannian manifold which is homotopy equivalent to T^n is, as we mentioned before, actually affine diffeomorphic to T^n. From Theorem B in the introduction we therefore know that $M = T^n \# \Sigma^n$ does not admit an integrable finite G-structure, when Σ^n is exotic. Nevertheless, we claim that we can find such a manifold M and a principal $\mathbf{Z}_k$-bundle $\pi: \overline{M} \to M$ in which $\overline{M}$ is the standard n-torus.

First note that the group of homotopy spheres θ_n in dimensions $n > 4$ is a finite group by Kervaire-Milnor [11], which acts freely on the smoothings of the underlying P.L.-structure on T^n; cf. Wall [13, §15A]. Choose an $n > 4$ such that θ_n is a cyclic group of order k, and let $\Sigma^n \in \theta_n$ be a generator for θ_n. Then $M = T^n \# \Sigma^n$ does not admit an integrable finite G-structure. However, we can construct a principal $\mathbf{Z}_k$-bundle $\pi: \overline{M} \to M$ in which $\overline{M}$ is diffeomorphic to T^n as follows. Define $t: \mathbf{R}^n \to \mathbf{R}^n$ by $t(x_1, x_2, \cdots, x_n) = (x_1 + 1/k, x_2, \cdots, x_n)$ and represent T^n as $\mathbf{R}^n/\mathbf{Z}^n$, where $\mathbf{Z}^n$ is generated by the translations $t_i: \mathbf{R}^n \to \mathbf{R}^n$, $i = 1, \cdots, n$, defined by $t_i(x_1, \cdots, x_i, \cdots, x_n) = (x_1, \cdots, x_i + 1, \cdots, x_n)$. Then t induces in an obvious way a free $\mathbf{Z}_k$-action on the manifold $\overline{M} = T^n \# \Sigma^n \# \cdots \# \Sigma^n$, k copies of Σ^n, with M as quotient. Since $\Sigma^n \in \theta_n$ has order k, $\overline{M}$ is diffeomorphic to T^n, and the quotient map $\pi: \overline{M} \to M$ is the principal bundle requested.

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